Last updated Feb 14 7:15 am

1. (Calc OK))A sample of Dl-13 (an isotope of Delerium) loses 99% of its radioactive matter in 199 hours. What is the half-life of Dl-13?
   (a) 4 hours
   (b) 6 hours
   (c) 30 hours
   (d) 100.5 hours
   (e) 143 hours
   (f) None of these

Direct Method

$50 = 100\left(\frac{1}{100}\right)^{t/199}$

$\frac{1}{2} = (.01)^{t/199}$

$\ln 1/2 = \frac{t}{199}\ln(.01)$

$t = \frac{199\ln(1/2)}{\ln(.01)} = 29.95258457$

$\boxed{\text{OR}}$

$P = Pe^{kt}$

$.01 = e^{199k}$

$k = \frac{\ln .01}{199}$

$\frac{1}{2} = e^{kt}$

$t = \frac{\ln(1/2)}{k} = \frac{199\ln 1/2}{\ln(.01)}$

2. In which of the following models is $\frac{dy}{dt}$ directly proportional to y?

so $\frac{dy}{dt} = ky$ Solution looks like $y = Ce^{kt}$ (or $y = \pm e^{kt+C_1}$)

   I $y = e^{kt} + C$ ✗
   II $y = Ce^{kt}$ ✓
   III $y = 28^{kt}$ $= e^{(\ln 28)kt}$ ✓
   IV $y = 3\left(\frac{1}{2}\right)^{3t+1}$ $= 3e^{(\ln\frac{1}{2})(3t+1)}$ ✓

   (a) I only
   (b) IIonly
   (c) I and II only
   (d) II and III only
   (e) II, III, and IV only
   (f) All four

$2\cdot Ce^{5k} = 3\cdot 2$

$3\cdot Ce^{k} = 2\cdot 3$

(get rid of C to find k)

3. (Calculator Active) The rate at which acreage is being consumed by a plot of kudzu is proportional to the number of acres already consumed at time t. If there are 2 acres consumed when $t = 1$ and 3 acres consumed when $t = 5$, how many acres will be consumed when $t = 8$?

$y = Ce^{kt}$ (1, 2) & (5, 3)

   (a) 3.750
   (b) 4.000
   (c) 4.066
   (d) 4.132
   (e) 4.600
   (f) None of these

$2\cancel{C}e^{5k} = 3\cancel{C}e^{k}$

$e^{4k} = 3/2$

$4k = \ln 3/2$

$k = \frac{\ln 3/2}{4}$

$\boxed{\text{STO}} \to k$

$2 = Ce^{(\ln 3/2)/4 \cdot t}$

$2 = Ce^{\ln(\frac{3}{2})^{1/4}}$

$2 = \left(\frac{3}{2}\right)^{1/4}\cdot C$

$C = 2\left(\frac{3}{2}\right)^{-1/4}$

$\boxed{\text{STO}} \to C$

$y(t) = 2\left(\frac{3}{2}\right)^{-1/4}\cdot e^{t\ln(3/2)/4}$

$Y_1(x) = Ce^{k*x}$ on your T.I

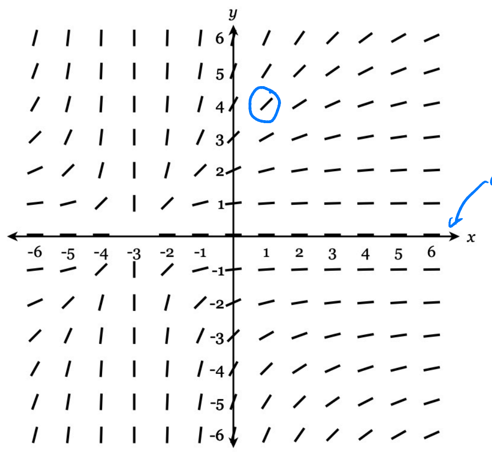
$Y_1(8) = 4.066209016$

4. Which of the following are separable differential equations? (There may be more than one)

of the Form $f\cdot g$

   (a) $\frac{dy}{dx} = \ln(x+y)$ NO
   (b) $\frac{dy}{dx} = e^{x+y}$ $= e^x \cdot e^y$ so yes
   (c) $\frac{dy}{dx} = x^2 + 3xy$ $= x(x+3y)$ so NO
   (d) $\frac{dy}{dx} = xy + 4x$ $= x(y+4)$ so yes
   (e) None of these

5. Select the differential equation that matches the given slope field.



0 slope at $y = 0$

div by 0 @ $x = -3$

(a) $\dfrac{dy}{dx} = -\dfrac{y}{x+3}$ $= -1$ @ $(1,4)$

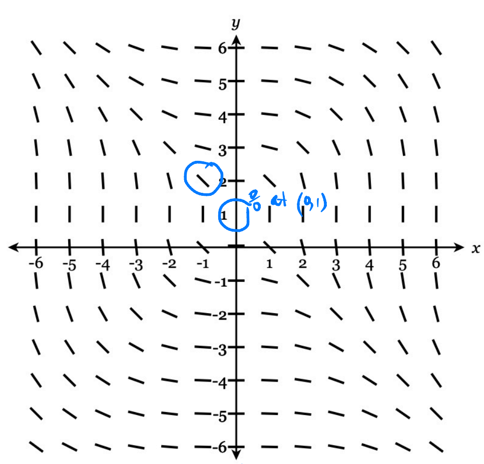
~~(b)~~ $\dfrac{dy}{dx} = (x+3)y^2$

~~(c)~~ $\dfrac{dy}{dx} = \dfrac{x+3}{y}$

$\boxed{\text{(d) } \dfrac{dy}{dx} = \dfrac{y^2}{(x+3)^2}}$ $= 1$ @ $(1,4)$

(e) None of these

6. Select the differential equation that matches the given slope field.



0 slope @ $x = 0$

~~(a)~~ $\dfrac{dy}{dx} = \dfrac{(y-1)^2}{x}$

(b) $\dfrac{dy}{dx} = -\dfrac{x}{y-1}$ @$(-1,2)$ $= \dfrac{+1}{2-1} = +1$

$\boxed{\text{(c) } \dfrac{dy}{dx} = -\dfrac{x^2}{(y-1)^2}}$ @ $(-1,2)$ $= \dfrac{-1}{2-1} = -1$

~~(d)~~ $\dfrac{dy}{dx} = -\dfrac{y-1}{x}$

(e) None of these

from 6.1 HW

7. Which of the following is a solution of the differential equation $xy' - 4y = x^{5}e^x$?
   (a) $y = 4x^5e^{2x}$
   (b) $y = 6e^{2x} - 7\sin 2x$
   (c) $y = x^4e^x$
   (d) $y = 5e^{-2x}$
   (e) $y = \ln x$
   (f) None of these

(a) $y' = 8x^5e^{2x} + 20x^4e^{2x}$
$xy' = 8x^6e^{2x} + 20x^5e^{2x}$
$xy' - 4y = 8x^6e^{2x} + 20x^5e^{2x} - 16x^5e^{2x}$
$= 8x^6e^{2x} + 4x^5e^{2x}$
Nope

(b) $y' = 12e^{2x} - 14\cos 2x$
$xy' = 12xe^{2x} - 14x\cos 2x$
$xy' - 4y = 12xe^{2x} - 14x\cos 2x - (24e^{2x} - 28\sin 2x)$
Nope

$$\boxed{\begin{aligned}&(c)\ y' = x^4e^x + 4x^3e^x\\ &xy' = x^5e^x + 4x^4e^x\\ &xy' - 4y = \underbrace{x^5e^x + 4\cancel{x^4e^x} - 4\cancel{x^4}e^x}_{x^5e^x\ \checkmark\ \text{yes!}}\end{aligned}}$$

(d) $y' = -10e^{-2x}$
$xy' = -10xe^{-2x}$
$xy' - 4y = -10xe^{-2x} - 5e^{-2x}$
Nope

(e) $y' = \frac{1}{x}$
$xy' = \frac{x}{x} = 1$
$xy' - 4y = 1 - \ln x$
nope

Practice at ΔMath

8. (a) (3 points) Consider the differential equation $\frac{dy}{dx} = xy^3$ with a particular solution $y = f(x)$ having an initial condition $y(-2) = -1$. Use the equation of the line tangent to the graph of f at the point $(-2, -1)$ in order to approximate the value of $f(-1.9)$.

$\left.\frac{dy}{dx}\right|_{(-2,-1)} = (-2)(-1)^3 = 2 \to$ slope
$y + 1 = 2(x+2) \to$ tangent line
$y = 2x + 3$
$y(-1.9) = -3.8 + 3 = -0.8$

Practice at ΔMath

(b) (3 points) Consider the differential equation $\frac{dy}{dx} = (x^2+3)(y-2)$ with a particular solution $y = f(x)$ having an initial condition $y(0) = -3$. Use the equation of the line tangent to the graph of f at the point $(0, -3)$ in order to approximate the value of $f(0.1)$.

$\left.\frac{dy}{dx}\right|_{(0,-3)} = (3)(-5) = -15 \to$ slope
$y + 3 = -15(x - 0) \to$ tangent line
$y = -15x - 3$
$y(0.1) = -1.5 - 3 = -4.5$

9. Find the general solution to the following differential equations, then find the particular solution using the initial condition.

(a) $\frac{dy}{dx} = \frac{x}{y}$, $y(1) = -2$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 = x^2 + C$

$y = \pm\sqrt{x^2+C}$

(Gen. Sol.)

at $(1,-2)$

$-2 = -\sqrt{1^2+C}$

$C = 4 - 1 = 3$

$y = -\sqrt{x^2+3}$

(particular Solution for $(1,-2)$)

(b) $\frac{dy}{dx} = -\frac{x}{y}$, $y(4) = 3$

$\int y\,dy = -\int x\,dx$

$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$

$y^2 = -x^2 + C$

$y = \pm\sqrt{C-x^2}$

at $(4,3)$

$3 = \sqrt{C-4^2}$

$9 = C-4^2$

$C = 9+16 = 25$

$y = \sqrt{25-x^2}$

(c) $\frac{dy}{dx} = \frac{y}{x}$, $y(2) = 2$

$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$

$\ln|y| = \ln|x| + C_1$

$|y| = e^{\ln|x|+C_1}$

$y = C e^{\ln|x|}$

$y = C|x|$

@ $(2,2)$

$2 = C|2|$

$C = 1$

$y = |x|$

(d) $\frac{dy}{dx} = 2xy$, $y(0) = -3$

$\int \frac{1}{y}\,dy = 2\int x\,dx$

$\ln|y| = x^2 + C_1$

$y = C e^{x^2}$

@ $(0,-3)$

$-3 = C e^{0^2}$

$C = -3$

$y = -3e^{x^2}$

(e) $\frac{dy}{dx} = (y+5)(x+2), \quad y(0) = -1$

$$\int \frac{1}{y+5}\, dy = \int x+2\, dx$$

$$\ln|y+5| = \frac{x^2}{2} + 2x + C_1$$

$$y+5 = C e^{x^2/2 + 2x}$$

$$y = C e^{(x^2+4x)/2} - 5$$

@ $(0,-1)$

$$-1 = C e^0 - 5$$

$$C = 4$$

$$y = 4e^{(x^2+4x)/2} - 5$$

(f) $\frac{dy}{dx} = \cos^2(y), \quad y(0) = 0$

$$\int \frac{1}{\cos^2 y}\, dy = \int 1\, dx$$

$$\int \sec^2 y = \int dx$$

$$\tan y = x + C$$

$$y = \arctan(x+C)$$

@ $(0,0)$

$$0 = \arctan(0+C)$$

$$C = 0$$

$$y = \arctan(x)$$

(g) $\frac{dy}{dx} = (\cos x)e^{y+\sin x}, \quad y(0) = 0$

$$dy = (\cos x)\, e^y \cdot e^{\sin x}\, dx$$

$$\int e^{-y}\, dy = \int e^{\sin x} \cos x\, dx$$

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x\, dx \end{array}}$$

$$-e^{-y} = \int e^u\, du$$

$$+e^{-y} = -e^{\sin x} + C$$

$$-y = \ln(C - e^{\sin x})$$

$$y = -\ln(C - e^{\sin x})$$

@ $(0,0)$

$$0 = -\ln(C - e^0)$$

$$0 = -\ln(C-1) \quad (= \ln 1 = -\ln 1)$$

$$1 = C - 1$$

$$1 = C - 1.$$

$$C = 2$$

$$y = -\ln(2 - e^{\sin x})$$

(h) $\frac{dy}{dx} = e^{x-y}, \quad y(0) = 2$

$dy = \frac{e^x}{e^y}\,dx$

$\int e^y\,dy = \int e^x\,dx$

$e^y = e^x + C$

$y = \ln(e^x + C)$

@ $(0,2)$

$2 = \ln(e^0 + C)$

$e^2 = 1 + C$

$C = e^2 - 1$

$y = \ln(e^x + e^2 - 1)$

(i) $\frac{dy}{dx} = -2xy^2, \quad y(1) = \frac{1}{4}$

$\int y^{-2}\,dy = -2\int x\,dx$

$-y^{-1} = -x^2 + C$

$\frac{1}{y} = x^2 + C$

$y = \frac{1}{x^2 + C}$

@ $(1, \frac{1}{4})$

$\frac{1}{4} = \frac{1}{1^2 + C}$

$4 = 1 + C$

$C = 3$

$y = \frac{1}{x^2 + 3}$

(j) $\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x}, \quad y(e) = 1$

$\int y^{-1/2}\,dy = 4\int \frac{\ln x}{x}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}\,dx$

$2y^{1/2} = 4\int u\,du$

$y^{1/2} = 2\frac{u^2}{2} + C_1$

$y^{1/2} = (\ln x)^2 + C_1$

$y = (\ln^2 x + C)^2$

or $((\ln x)^2 + C)^2$

@ $(e,1)$

$y = [(\ln e)^2 + C]^2$

$1 = 1 + C$

$C = 0$

$y = [(\ln x)^2]^2$

$y = (\ln x)^4$

or

$y = \ln^4 x$

10. Find the solution of the differential equation $\frac{dy}{dt} = ky$ that satisfies the given conditions.

(a) $y(0) = 50$ and $y(5) = 100$

$y = Ce^{kt}$

using $(0, 50)$

$50 = Ce^0$

$C = 50$

using $(5, 100)$

$100 = 50e^{5k}$

$2 = e^{5k}$

$k = \frac{\ln 2}{5}$

$y(t) = 50e^{t \ln 2/5}$

for what t is $y(t) = 75$?

$75 = 50e^{t(\ln 2)/5}$

$\frac{3}{2} = e^{t(\ln 2)/5}$

$\ln\frac{3}{2} = t(\ln 2)/5$

$t = \frac{5\ln(3/2)}{\ln 2} = \log_2\left(\frac{3}{2}\right)^5$

≈ 2.924812504

(b) The graph of y passes through $(1, 55)$ and $(10, 30)$

$\frac{30}{55} = \frac{Ce^{10k}}{Ce^{k}} \Bigg\} \Rightarrow \frac{6}{11} = e^{9k}, \quad k = \frac{\ln(6/11)}{9}$ $\boxed{\text{STO}\rightarrow}$ K

$55 = Ce^{\frac{\ln(6/11)}{9}} = Ce^{\ln(6/11)^{1/9}} = C\left(\frac{6}{11}\right)^{1/9}$

$C = 55 \cdot \left(\frac{6}{11}\right)^{-1/9}$ $\boxed{\text{STO}\rightarrow}$ C

$y = 55\left(\frac{11}{6}\right)^{1/9} e^{t\left(\frac{\ln 6/11}{9}\right)}$

$y(5) = Ce^{5K}$

$= 42.01133634$

11. Write and find a general solution of the differential equation that describes this statement: The rate of change of G with respect to to t is proportional to $50 - t$.

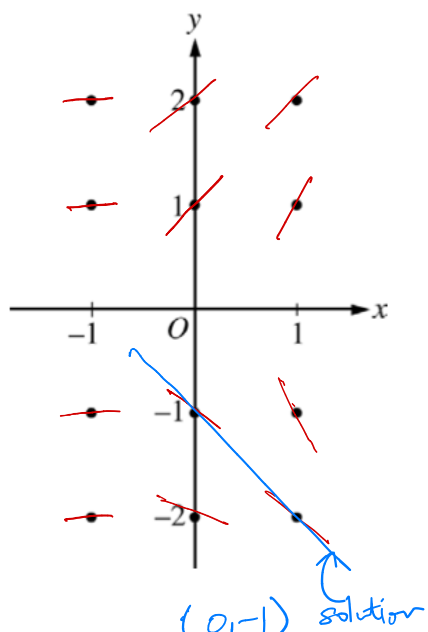
$$\frac{dG}{dt} = k(50 - t)$$

General Solution: $\int dG = k\int 50 - t \, dt$

$G = k\left(50t - \frac{t^2}{2}\right) + C$

12. (2010B AB 5 - No Calc) Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

(a) (3 points) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.



x	y	$\frac{dy}{dx} = \frac{x+1}{y}$
-1	2	0
-1	1	0
0	2	$\frac{1}{2}$
0	1	1
0	-1	-1
0	-2	$-1/2$
1	2	$2/2 = 1$
1	1	$2/1 = 2$
1	-1	$2/-1 = -2$
1	-2	$2/-2 = -1$
-1	0	

(b) (1 point) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane for which $y \neq 0$. Describe all points in the xy-plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

$$\frac{x+1}{y} = -1$$

$$-y = x+1$$

$$y = -x-1$$

all points $(x,y) \in \mathbb{R}_2$ such that $y = -x-1$

$$\{(x,y) \in \mathbb{R}_2 \mid y = -x-1\}$$

(c) (5 points) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$

$$\frac{dy}{dx} = \frac{x+1}{y}$$

$$\int y\,dy = \int x+1\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + x + C_1$$

$$y^2 = x^2 + 2x + C$$

$$y = \pm\sqrt{x^2+2x+C}$$

$(0,-2)$:

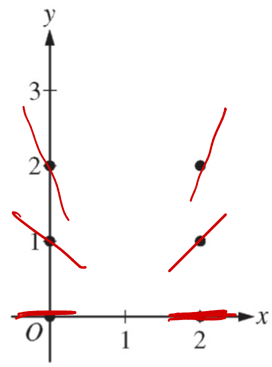
$$-2 = -\sqrt{0^2+0+C}$$

$$C = 4$$

$$y = -\sqrt{x^2+2x+4}$$

13. (2016 AB 4 - No Calc) Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) (2 points) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



x	y	$y^2/x-1$
0	2	$4/-1 = -4$
0	1	$1/-1 = -1$
0	0	$0/-1 = 0$
2	2	$4/2-1 = 4$
2	1	$1/2-1 = 1$
2	0	$0/2-1 = 0$

(b) (2 points) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Us your equation to approximate $f(2.1)$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}\Big|_{(2,3)} = \frac{3^2}{2-1} = 9 \text{ slope}$$

$$y - 3 = 9(x-2)$$

$$y = 9(x-2) + 3$$

$$y(2.1) = 9(2.1-2)+3 = .9 + 3 = 3.9$$

(c) (5 points) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}$$

$$\int y^{-2}\,dy = \int \frac{1}{x-1}\,dx$$

$$-y^{-1} = \ln|x-1| + C_1$$

$$-\frac{1}{y} = \ln|x-1| + C$$

$(2,3)$

$$-\frac{1}{3} = \ln|2-1| + C$$

$$-\frac{1}{3} = 0 + C$$

$$C = -\frac{1}{3}$$

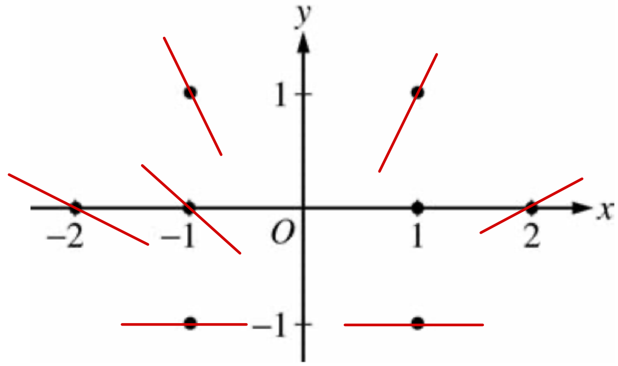
$$-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$$

$$-\frac{1}{y} = \frac{3\ln|x-1| - 1}{3}$$

$$y = \frac{3}{1 - 3\ln|x-1|}$$

14. (2006 AB 5 - No Calc) Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$ where $x \neq 0$

(a) (2 points) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.



x	y	$\frac{1+y}{x}$
-1	1	$2/-1 = -2$
-1	0	$1/-1 = -1$
-1	-1	$0/-1 = 0$
1	1	$2/1 = 2$
1	0	$1/1 = 1$
1	-1	$0/1 = 0$
-2	0	$1/-2 = -1/2$

(b) (7 points) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain..

$$\frac{dy}{dx} = \frac{1+y}{x}$$

$$\int \frac{1}{1+y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|1+y| = \ln|x| + C$$

$$|1+y| = e^{\ln|x| + c} = Ce^{\ln x} = C|x|$$

$$1 + y = \pm C|x|$$

$$y = \pm c|x| - 1$$

@ $(-1, 1)$

$$1 = C - 1$$

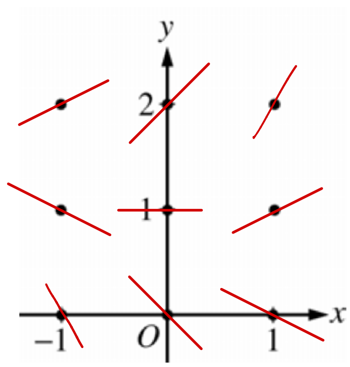
$$c = 2$$

$$y = 2|x| - 1$$

Domain: $(-\infty, 0)$

15. (2007B AB 5 - No Calc) Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}x + y - 1$.

(a) (2 points) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.



x	y	$\frac{1}{2}x+y-1$
-1	2	$-\frac{1}{2}+2-1 = \frac{1}{2}$
-1	1	$-\frac{1}{2}+1-1 = -\frac{1}{2}$
-1	0	$-\frac{1}{2}+0-1 = -\frac{3}{2}$
0	2	$0+2-1 = 1$
0	1	$0+1-1 = 0$
0	0	$0+0-1 = -1$
1	2	$\frac{1}{2}+2-1 = \frac{3}{2}$
1	1	$\frac{1}{2}+1-1 = \frac{1}{2}$
1	0	$\frac{1}{2}+0-1 = -\frac{1}{2}$

(b) (3 points) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Describe the region in the xy-plane in which all solution curves to the differential equation are concave up.

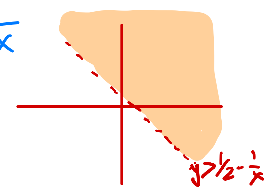
$$\frac{d}{dx}\left(\frac{1}{2}x+y-1\right)$$

$$= \frac{1}{2} + \frac{dy}{dx} = \frac{1}{2} + \frac{1}{2}x + y - 1$$

must be positive to be concave up

so $\frac{1}{2} + \frac{1}{2}x + y - 1 > 0$

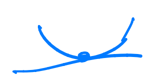
$y > \frac{1}{2} - \frac{1}{x}$



(c) (2 points) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. Does f have a relative minimum, a relative maximum, or neither at $x = 0$? Justify your answer.

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1}{2}(0) + 1 - 1 = 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 1 > 0 \text{ concave up}$$



(0, 1) is a rel. min.

(d) (2 points) Find the values of the constants m and b, for which $y = mx + b$ is a solution to the differential equation.

$y = mx + b$

$\frac{dy}{dx} = (m) + (0)x$

$\frac{dy}{dx} = \frac{1}{2}x + (mx + b) - 1$ (y)

$\frac{dy}{dx} = x\left(\frac{1}{2} + m\right) + (b - 1)$

match coeff

$\frac{1}{2} + m = 0$, $m = \frac{-1}{2}$

$b - 1 = m$, $b = m + 1 = \frac{-1}{2} + 1 = \frac{1}{2}$